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Deformations of the canonical commutation relations

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Deformations of the canonical commutation relations which have the effect of altering the spectrum of a standard Hamiltonian, bilinear in creation and annihilation operators are described. The problem of going over from an eigenvalue situation, as is the case in the vast majority of papers in the literature, to a theory with time evolution is discussed, and a special example with deformation parameter an Nth root of unity is constructed which possesses a consistent time evolution. This work is an account of some recent studies of associative deformations of the Heisenberg algebra of several creation and annihilation algebras, with Jean Nuyts of the University of Mons, Hainaut, together with some observations of my own concerning the difficulty of implementing time evolution in a quantum group context. It builds on earlier work with Cosmas Zachos (Argonne National Laboratory, USA), which in turn is related to work of Manin, and Wess, Zumino and collaborators. The main idea is that, if quantum groups have any role in physics, then they must manifest themselves at the level of the basic rules of quantisation.

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1. Alternative oscillator quantisation

In ref. [1] the object of the exercise is to take a standard Hamiltonian,

$$\mathcal{H}=\sum_{j=1}^{N}(1-d_j)x(j)x(-j)\ ,$$

bilinear in the creation operators $x(j)$ and destruction operators $x(-j), j=1, ..., N$, and construct an alternative quantisation such that the spectrum is different from the one obtained by canonical quantisation. We consider a set of rules for the operators $x(j)$, $0<j\leqslant N$, which are considered as creation-like operators, with destruction operators $x(-j)$ which annihilate the vacuum state. We extend the notion of normal ordering to require that a product $x(j_1)x(j_2)\cdots(j_n)$, with $-N\leqslant j_i\leqslant N$ is written in decreasing order of its indices. The rules we propose implement associativity of this product, and guarantee that all ways of re-order-

ing a given product to put it in normal form are equivalent. The most general parametrisation we have found within the requirement that no additional operators other than a single central term which commutes with everything are introduced is as follows:

$$x(j)x(k)=r_{j,k}x(k)x(j) \qquad \text{for } k>j>0\,,$$

$$x(-j)x(-k)=c_{j,k}x(-k)x(-j) \qquad \text{for } j>k>0\,,$$

$$x(-j)x(k)=r_{j,k}^{-1}x(k)x(-j) \qquad \text{for } k>j>0\,,$$

$$x(-j)x(k)=c_{j,k}^{-1}x(-k)x(-j) \qquad \text{for } j>k>0\,,$$

$$x(-j)x(j)=\sum_{k=1}^{j}f_{j,k}x(k)x(-k)+d_j\,. \tag{1.1}$$

The parameters $f_{j,k}$ are given by

$$f_{j,k}=\frac{d_j}{d_k}\frac{1}{(r_{k,j}c_{j,k}-1)} \quad (k\neq j)\,,$$

$$f_{j,j}=\frac{1}{r_{j,j+1}c_{j+1,j}}\,. \tag{1.2}$$

The parameters $r_{j,k}$ and $c_{k,j}$, defined for $k>j>0$, may be regarded as the superdiagonal and the subdiagonal entries in the same $N\times N$ matrix. There are $(N-1)(N-2)/2$ relations between them which prescribe all $c_{k,j}$ except $c_{j+1,j}$, $j<N$, in terms of $r_{j,k}$. These are as follows:

$$c_{k,j}r_{j,k}=r_{j,j+1}c_{j+1,j} \quad \text{for } k\neq j+1, j<N\,. \tag{1.3}$$

This is a slight, but nevertheless crucial extension of the deformed algebra of Schirrmacher, Wess and Zumino [2]. Now consider the quadratic Hamiltonian

$$H=\sum_{j=1}^{N}(1-d_j)x(j)x(-j)\,.$$

With the usual quantisation scheme, the spectrum of this hamiltonian is simply $\sum_j(1-d_j)n_j$ for occupation numbers n_j. In the case when the quantisation is done with the associative rules of eq. (1.1), we obtain

$$Hx(j)-d_jx(j)H=(1-d_j)x(j)\,. \tag{1.4}$$

This means that the state $\Pi_j\, x(j)^{n_j}$ has spectrum $1-\Pi_j\, d_j^{n_j}$ instead. From the point of view of refs. [3,4] this result is something of an illusion, as there exists a highly nonlinear transformation of variables which represent $x(j)$, $x(-j)$ in terms of operators satisfying the usual canonical commutation relations. However, the Hamiltonian will also be highly nonlinear when expressed in terms of the usual creation and destruction operators, so the issue of which is the preferable form is an aesthetic judgement.

2. Inclusion of neutral operators

One evident direction to proceed, when alternative associative deformations of the canonical commutation relations are sought, is the search for a set of relations which will interpolate between bosons (for $q=1$) and fermions ($q=-1$). Now since the fermionic relations admit a finite dimensional representation, a requirement will be the necessity for the interpolating algebras to possess finite dimensional representations, at least for specific values of q. It will turn out that these correspond to Nth roots of unity.

In all deformations discussed in the literature of which I am aware, the assumption is made that a creation operator $b^\dagger(a)$ and an annihilation operator $b(k)$ which are not conjugates of each other (i.e., when, say $a\neq k$), always quommute with each other with no inhomogeneous term. Jean Nuyts and I [5] analysed a system in which this is no longer true. It might be thought that any such terms might be transformed away by change of basis; indeed this is the case classically, but is not so when deformations are considered, and it is necessary to introduce such terms to obtain finite dimensional representations in the general case. For a set of operators which fulfill among themselves quommutation relations the consistency requirements imposed by associativity on the triple products of the operators are the braiding relations

$$(A*B)*C=A*(B*C)\ . \tag{2.1}$$

First consider a set of rules for a set of N operators $b^\dagger(a)$, $1\leqslant a\leqslant N$, which are considered as creation operators with an equal number of destruction operators $b(j)$. Suppose we also introduce a set of N^2 extra neutral operators, $y(j,a)$, $1\leqslant j\leqslant N$, $1\leqslant a\leqslant N$, which are considered as neither creation nor annihilation operators, and which commute amongst themselves. The total set thus contains $2N+N^2$ operators, namely $b(j)$, $b^\dagger(a)$ and $y(j,a)$.

Now extend the notion of normal ordering by requiring

(a) that a product of creation operators $b^\dagger(a_1)b^\dagger(a_2)\cdots b^\dagger(a_n)$, with $1\leqslant a_i\leqslant N$, is written in decreasing order of its indices;

(b) that a product of annihilation operators $b(j_1)b(j_2)\cdots b(j_n)$, with $1\leqslant j_i\leqslant N$, is written in increasing order of its indices;

(c) that destruction operators appear on the right of creation operators;

(d) that the neutral operators $y(j,a)$ are at the left of the annihilation operators and at the right of creation operators; since they commute their own order is irrelevant.

The rules we propose implement associativity of these products, and guarantee that all ways of re-ordering a given product to put it in normal form are equivalent. We have found very general parametrisations under the hypothesis of a quadratic quommutator algebra which can be summarized as follows: allowed fully consistent quantum algebras are given by the set of quommutators

$$b^\dagger(a)b^\dagger(b)=\epsilon(n_a/n_b)b^\dagger(b)b^\dagger(a)\ , \tag{2.2a}$$

$$b(j)b(k)=\epsilon(m_k/m_j)b(k)b(j)\ , \tag{2.2b}$$

$$b(j)b^\dagger(a)=\phi m_j n_a b^\dagger(a)b(j)+y(j,a)\ , \tag{2.2c}$$

$$b(j)y(k,a)=\epsilon\phi m_k n_a y(k,a)b(j)\ , \tag{2.2d}$$

$$y(j,a)b^\dagger(b)=\epsilon\phi m_j n_a b^\dagger(b)y(j,a)\ , \tag{2.2e}$$

$$y(j,a)y(k,b)=y(k,b)y(j,a)\ , \tag{2.2f}$$

$$\epsilon^2=1\ , \tag{2.2g}$$

where $\phi\neq 0$ can be renormalized to unity. This set of relations admits a representation by infinite dimensional matrices, extending the work of Manin and others [6,7].

Let the b's be infinite matrices with nonzero elements on the diagonal just above the main diagonal only,

$$b(j)_{p,p+1}=B(j)_p \quad \text{for } p=1,2,\ldots\ . \tag{2.3}$$

Imposing (2.2a), one proves easily that

$$B(j)_p=\beta_j X_p m_j^{p-1} \quad \text{for } p=1,2,\ldots\ . \tag{2.4}$$

By renormalizing the b's, the factors β can be normalized to one.

In an analogous fashion the infinite matrices representing the $b^\dagger$'s have nonzero elements just below the main diagonal and are of the form

$$b^\dagger(a)_{p+1,p}=B^\dagger(a)_p \quad \text{for } p=1,2,\ldots\ , \tag{2.5}$$

with, to satisfy (2.2b) and with a suitable normalization,

$$B^\dagger(a)_p=X_p^\dagger n_a^{p-1}\ , \tag{2.6}$$

where the $X^\dagger$'s are the conjugates of the X's if one imposes that the operators $b^\dagger$ be the adjoints of the b's.

From the expressions (2.3)–(2.6) the y's computed from (2.2c) are diagonal with diagonal elements

$$y(j,a)_{1,1}=Z_1\ , \tag{2.7a}$$

$$y(j,a)_{p,p}=m_j^{p-1}n_a^{p-1}(Z_p-Z_{p-1}) \quad \text{for } p=2,3,\ldots\ , \tag{2.7b}$$

where, for convenience, we have written

$$Z_p = X_p X_p^\dagger . \tag{2.8}$$

Since they are diagonal the y's formally commute.

Using (2.2d), (2.2e) (for $\epsilon = 1$) and the last equations (2.7a), (2.7b), the Z's are constrained to fulfill the relations

$$Z_2 = 2Z_1 , \tag{2.9a}$$

$$Z_{p+1} = 2Z_p - Z_{p-1} \quad \text{for } p = 2, 3, \dots . \tag{2.9b}$$

The solution of these recurrence equations is given in terms of the free parameter Z_1 as

$$Z_p = pZ_1 . \tag{2.10}$$

The X_p are determined by (2.8) up to arbitrary factors. These arbitrary factors become phase factors if the $\bar{b}$'s are chosen to be the adjoints of the b's.

3. Alternative quantum algebra

We now introduce extra relations which, in some sense, impose a symmetry between the indices of the creation operators on one hand and between the indices of the annihilation operators on the other, *as seen from the point of view of the y's*. Specifically

$$y(j, a) y(k, b) = v_{jakb} y(j, b) y(k, a) \quad \text{for some } v_{jakb} . \tag{3.1}$$

Then fully consistent quantum algebras are given by the following sets of quommutators:

$$b^\dagger(a) b^\dagger(b) = \epsilon(n_a / n_b) b^\dagger(b) b^\dagger(a) , \tag{3.2a}$$

$$b(j) b(k) = \epsilon(m_k / m_j) b(k) b(j) , \tag{3.2b}$$

$$b(j) b^\dagger(a) = \sigma m_j n_a b^\dagger(a) b(j) + y(j, a) , \tag{3.2c}$$

$$b(j) y(k, a) = \phi m_k n_a y(k, a) b(j) , \tag{3.2d}$$

$$y(j, a) b^\dagger(b) = \phi m_j n_a b^\dagger(b) y(j, a) , \tag{3.2e}$$

$$y(j, a) y(k, b) = y(k, b) y(j, a) , \tag{3.2f}$$

$$y(j, a) y(k, b) = \epsilon y(j, b) y(k, a) , \tag{3.2g}$$

$$y(j, a) b(k) = \epsilon y(k, a) b(j) , \tag{3.2h}$$

$$b^\dagger(a) y(j, b) = \epsilon b^\dagger(b) y(j, a) , \tag{3.2i}$$

$$\epsilon^2 = 1 , \tag{3.2j}$$

where $\phi \neq 0$ or $\sigma \neq 0$ can be renormalized to unity but not ϕ and σ at the same time. The essential parameter ϕ/σ is invariant under renormalisation.

The proofs will be omitted here.

It should be stressed that eqs. (3.2) imply that, if it is invertible, $y(j, a)$, considered as a formal matrix in j and a, is of rank one and hence can be written in a factorized form in terms of $2N$ commuting operators $y_1(j)$ and $y_2(a)$,

$$y(j, a) = y_1(j) y_2(a) , \tag{3.3}$$

with $\epsilon = 1$ and with the symmetry properties

$$y_1(j) b(k) = y_1(k) b(j) , \quad b^\dagger(a) y_2(b) = b^\dagger(b) y_2(a) . \tag{3.4}$$

This algebra possesses both some finite and infinite dimensional representations.

Representations of the quantum algebra (3.2) can be obtained with infinite dimensional matrices but also with finite $M \times M$, $M > 2$, matrices. We here present solutions for three types of matrices, namely the cases (a), (b) and (c) below:

(a) Infinite matrices b ($b^\dagger$) which have nonzero elements on the diagonal just above (below) the main diagonal only,

$$b(j)_{p,p+1} = X_p m_j^{p-1} , \quad b^\dagger(a)_{p+1,p} = X_p^\dagger n_a^{p-1} , \quad \text{for } p = 1, 2, \dots . \tag{3.5}$$

(b) Finite $M \times M$ matrices b ($b^\dagger$) which have nonzero elements on the diagonal just above (below) the main diagonal only, i.e., (3.5) with $p = 1, 2, \dots, M-1$.

(c) Finite $M \times M$ matrices b ($b^\dagger$) which have nonzero elements on the diagonal just above (below) the main diagonal only, i.e. (3.5), together with a nonzero element in the lower left (upper right) corner,

$$b(j)_{M,1} = X_M m_j^{M-1} , \quad b^\dagger(a)_{1,M} = X_M^\dagger n_a^{M-1} . \tag{3.6}$$

3.1. REPRESENTATIONS IN CASE (a)

When the infinite matrices b and $b^\dagger$ have their nonzero elements as in (3.5), the y's of (2.2c) can be written, as expected, in the form (3.2) with the diagonal matrices $y_1(j)$ and $y_2(a)$ (with some obvious arbitrary choices),

$$y_1(j)_{p,p} = m_j^{p-1} \qquad \text{for } p = 1, 2, \dots , \tag{3.7}$$

$$y_2(a)_{1,1} = Z_1 , \tag{3.8a}$$

$$y_2(a)_{p,p} = n_a^{p-1}(Z_p - \sigma Z_{p-1}) \quad \text{for } p = 2, 3, \dots . \tag{3.8b}$$

These solutions satisfy the symmetry relations (3.4) automatically.

The remaining equations to be solved are (2.2a), (2.2b), which become

$$Z_2 = (\sigma + \phi) Z_1 , \tag{3.9a}$$

$$Z_{p+1} = (\sigma + \phi) Z_p - \sigma\phi Z_{p-1} \quad \text{for } p = 2, 3, \dots, M-2 . \tag{3.9b}$$

Equations (3.9a), (3.9b) allow the determination of all the Z's from Z_1,

$$Z_p = \frac{\sigma^p - \phi^p}{\sigma - \phi} Z_1 \quad \text{for } p=2, 3, \dots , \tag{3.10}$$

and there is no restriction a priori on ϕ or σ.

3.2. REPRESENTATIONS IN CASE (b)

When the finite $M \times M$ matrices b and $b^\dagger$ have their nonzero elements as in (3.5), the y's of (2.2c) can again be written, as expected, in the form (3.3) with the diagonal matrices $y_1(j)$ and $y_2(a)$ as in (3.7) up to $p=M-1$, except that now

$$y_2(a)_{M,M} = -n_a^{N-1} \sigma Z_{M-1} . \tag{3.11}$$

The remaining equations (3.8), which are valid up to $p=M-2$, have to be supplemented by a last equation

$$0 = (\sigma + \phi) Z_{M-1} - \sigma\phi Z_{M-2} . \tag{3.12}$$

This last equation (3.12) is a consistency requirement

$$\frac{\sigma^M - \phi^M}{\sigma - \phi} = 0 . \tag{3.13}$$

Once, say, σ is renormalized to one, ϕ has to be a Mth root of unity satisfying

$$\phi^{M-1} + \phi^{M-2} + \cdots + \phi + 1 = 0 . \tag{3.14}$$

Note that the matrices b, $b^\dagger$ are nilpotent.

3.3. REPRESENTATIONS IN CASE (c)

When the b ($b^\dagger$) are represented by $M \times M$ matrices with nonzero elements on the diagonal above (below) the main diagonal but also with a nonzero element $(M, 1)$ [$(1, M)$], one obtains first that

$$m_j^M = \bar{m} , \quad n_a^M = \bar{n} , \tag{3.15}$$

with $\bar{m}$ and $\bar{n}$ independent of the indices j and a, respectively. (We may assume that the determinants of the b ($b^\dagger$) are nonzero.)

As expected, the y's assume the form (3.5), with, arbitrary choices being made as before, the diagonal matrices y_1,

$$y_1(j)_{p,p} = m_j^{p-1} , \tag{3.16a}$$

and y_2,

$$y_2(a)_{1,1}=Z_1-\bar{m}\bar{n}\sigma Z_m\,, \tag{3.16b}$$

$$y_2(a)_{p,p}=n_a^{p-1}(Z_p-\sigma Z_{p-1}) \quad \text{for } p=2,3,...,M\,. \tag{3.16c}$$

One has finally to solve the last equations (2.2a), (2.2b),

$$Z_2-(\sigma+\phi)Z_1+\bar{m}\bar{n}\sigma\phi Z_M=0\,, \tag{3.17a}$$

$$Z_{p+1}-(\sigma+\phi)Z_p+\sigma\phi Z_{p-1}=0 \quad \text{for } p=2,3,...,M-1\,, \tag{3.17b}$$

$$Z_1-\bar{m}\bar{n}(\sigma+\phi)Z_M+\bar{m}\bar{n}\sigma\phi Z_{M-1}=0\,. \tag{3.17c}$$

These equations allow the determination of all the Z's provided one consistency relation is satisfied, which reads

$$(\sigma^M\bar{m}\bar{n}-1)(\phi^M\bar{m}\bar{n}-1)=0\,. \tag{3.18}$$

The solution

$$\bar{m}\bar{n}=1/\sigma^M\,, \tag{3.19}$$

which implies

$$Z_p=\sigma^{p-}Z_1 \quad \text{for } p=2,3,...,M\,, \tag{3.20}$$

should be rejected since by (3.14) all the y's are forced to be zero. The solution

$$\bar{m}\bar{n}=1/\phi^M \tag{3.21}$$

implies

$$Z_p=\phi^{p-1}Z_1 \quad \text{for } p=2,3,...,M\,, \tag{3.22}$$

and generates a perfectly valid representation. For this representation the non-zero matrix elements of y_2 are

$$y_2(a)_{p,p}=(n_a\phi)^{p-1}(1-\sigma/\phi)Z_1 \quad \text{for } p=1,2,...,M\,, \tag{3.23}$$

where we recall that, in general, ϕ can be put to one by renormalisation [see the discussion following (2.5)].

4. The problem of time

The problem of the incorporation of time dependence into quantum group operators, i.e. the introduction of a time dependence into the representations, is more difficult than has been generally realised. After all there have already appeared papers which purport to solve the more difficult problem of setting up a gauging of a quantum group, i.e., of introducing a quovariant derivative. In my view none of these present a satisfactory solution to the problem. On a naive level, one should expect that time evolution should be realised, as in Lie theory, by a similarity transformation of the operators. This clearly respects the relations in a bilinear algebra. On the other hand, the time evolution in a standard quantum mechanical system is prescribed also by the commutators with the Hamiltonian; these relations are difficult to maintain. In fact, as far as I am aware, the example of the final section is the only one which is fully consistent. It makes use of representations of the second type of quantum algebra with neutral elements described in section 3.

5. How to quantise $(\mathrm{d}^3/\mathrm{d}t^3)x=x$

This section address the question as to how one might quantise dynamical equations of the form

$$\partial^n x/\partial t^n = (-1)^n x . \tag{5.1}$$

Consider the Hamiltonian

$$\mathscr{H} = \sum_{i=1}^{n} x_i p_{i+1} , \quad p_0 \equiv p_n .$$

The classical equations of motion for the coordinate x_j are

$$\dot{x}_j = \{\mathscr{H}, x_j\} = -x_{j-1} . \tag{5.2}$$

These imply that x_j, $\forall j$ satisfy eq. (5.1). This system may be quantised in the standard fashion, with canonical commutation rules

$$[x_i, x_j] = 0 , \quad [p_i, p_j] = 0 , \quad [x_i, p_j] = \mathrm{i}\hbar\delta_{ij} , \tag{5.3}$$

and with the replacement of (5.2) by

$$\dot{x}_j = [H, x_j] = -x_{j-1} . \tag{5.4}$$

The spectrum, at least for odd n is unbounded.

Quantisation with quommutators. It is sufficient to consider the case where $n=3$,

$$\mathscr{H} = x_1 p_2 + x_2 p_3 + x_3 p_1 . \tag{5.5}$$

It is easier to start with the representation

$$x_1 = \begin{pmatrix} \cdot & \mathrm{e}^{-\omega t} & \cdot \\ \cdot & \cdot & \mathrm{e}^{-t} \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad x_2 = \begin{pmatrix} \cdot & \mathrm{e}^{-\omega t} & \cdot \\ \cdot & \cdot & \omega\mathrm{e}^{-t} \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad x_3 = \begin{pmatrix} \cdot & \mathrm{e}^{-\omega t} & \cdot \\ \cdot & \cdot & \omega^2\mathrm{e}^{-t} \\ \cdot & \cdot & \cdot \end{pmatrix}, \tag{5.6}$$

$$p_1=\begin{pmatrix} \cdot & \cdot & \cdot \\ e^{\omega t} & \cdot & \cdot \\ \cdot & -\omega e^{t} & \cdot \end{pmatrix}, \quad p_2=\begin{pmatrix} \cdot & \cdot & \cdot \\ e^{\omega t} & \cdot & \cdot \\ \cdot & -e^{t} & \cdot \end{pmatrix}, \quad p_3=\begin{pmatrix} \cdot & \cdot & \cdot \\ e^{\omega t} & \cdot & \cdot \\ \cdot & -\omega^2 e^{t} & \cdot \end{pmatrix}, \tag{5.7}$$

with $\omega^3=1$. These satisfy the quommutation relations

$$x_1x_2=\omega x_2x_1\,, \quad x_1x_3=\omega^2x_3x_1\,, \quad x_2x_3=\omega x_3x_2\,,$$
$$p_1p_2=\omega p_2p_1\,, \quad p_1p_3=\omega^2p_3p_1\,, \quad p_2p_3=\omega p_3p_2\,,$$
$$x_1p_1-\omega^2p_1x_1=I\,, \quad x_2p_2-\omega^2p_2x_2=I\,, \quad x_3p_3-\omega^2p_3x_3=I\,,$$
$$x_1p_2-\omega p_2x_1=A\,, \quad x_2p_3-\omega p_3x_2=A\,, \quad x_3p_1-\omega p_1x_3=A\,,$$
$$x_1p_3-p_3x_1=B\,, \quad x_2p_1-p_1x_2=B\,, \quad x_3p_2-p_2x_3=B\,.$$

In the above, besides the identity operator I, two other operators, A, B, have occurred. These operators commute, $AB-BA=0$, since they are realised by diagonal matrices, as is the Hamiltonian,

$$A=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \omega^2 & \cdot \\ \cdot & \cdot & \omega \end{pmatrix}, \quad B=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \omega & \cdot \\ \cdot & \cdot & \omega^2 \end{pmatrix}, \quad \mathscr{H}=\begin{pmatrix} 3 & \cdot & \cdot \\ \cdot & -3 & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}. \tag{5.8}$$

Also

$$x_jI-Ix_j=0\,, \quad x_jA-\omega Ax_j=0\,, \quad x_jB-\omega^2Bx_j=0\,, \quad \forall j\,. \tag{5.9}$$

The Poisson bracket relations are replaced by the quommutators

$$\dot{x}_1=\tfrac{1}{3}(\mathscr{H}x_1-\omega^2x_1\mathscr{H})=-3\omega x_3\,,$$
$$\dot{x}_2=\tfrac{1}{3}(\mathscr{H}x_2-\omega^2x_2\mathscr{H})=-3\omega x_1\,,$$
$$\dot{x}_3=\tfrac{1}{3}(\mathscr{H}x_3-\omega^2x_3\mathscr{H})=-3\omega x_2\,. \tag{5.10}$$

The factor $1/3$ is introduced as an effective $\hbar$. Now consider, under the assumption that t acts as an ordinary variable,

$$(\mathrm{d}/\mathrm{d}t)(x_1x_2-\omega x_2x_1)=\dot{x}y+x\dot{y}-\omega(\dot{y}x+y\dot{x})$$
$$=-\omega(x_3x_2+x_1^2)-\omega(x_1^2+x_2x_3)\,. \tag{5.11}$$

This expression does not vanish for general quommutators (5.6), (5.7). However, it does vanish in the specific representation given. This is true for all derivatives of these quommutators; in the given representation they hold consistently. The representation given may be generalised to all integral n. It cannot be thought of as a faithful representation of an abstract quommutator scheme as clearly not all the operators are linearly independent. There are the relations

$$x_1 + \omega x_2 + \omega^2 x_3 = 0 ,$$
$$p_1 + \omega^2 p_2 + \omega p_3 = 0 . \qquad (5.12)$$

Nevertheless it provides a consistent scheme for quantisation with quommutators which possess a simple evolution in time. Note also that all quommutators of any "position" operator with any "momentum" operator contains a central term. The time dependence of the solution is consistent with ordinary differentiation.

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